NPS55-90-17

NAVAL POSTGRADUATE SCHOOL

Monterey, California

MODELING AND SIMULATION OF RANDOM SHAPES BY SCULPTURED WRAPAROUND: PRELIMINARY REPORT

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August 1990

Prepared for:
Naval Postgraduate School,
Monterey, CA 93955

NAVAL POSTGRADUATE SCHOOL,
MONTEREY, CALIFORNIA

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This report was prepared in conjunction with research funded under the Naval Postgraduate School Research Council Research Program.

This report was prepared by:

ssified

/ Classification of this page

REPORT DOCUMENTATION PAGE

oort Security Classification UNCLASSIFIED	1b Restrictive Markings
urity Classification Authority	3 Distribution Availability of Report Approved for public release; distribution is unlimited
classification/Downgrading Schedule	
orming Organization Report Number(s) NPS55-90-17	5 Monitoring Organization Report Number(s)
ne of Performing Organization Postgraduate School — 6b Office Symbol *(If Applicable)* OR	7a Name of Monitoring Organization
lress *(city, state, and ZIP code)* erey, CA 93943-5000	7b Address *(city, state, and ZIP code)*
ne of Funding/Sponsoring Organization aval Postgraduate School — 8b Office Symbol *(If Applicable)*	9 Procurement Instrument Identification Number 0&MN, Direct Funding
lress *(city, state, and ZIP code)* Monterey, California Monterey, CA 93943	10 Source of Funding Numbers

Program Element Number	Project No	Task No	Work Unit Accession No

le *(Include Security Classification)* Modeling and Simulation of Random Shapes by Sculptured Wraparound:
ninary Report

rsonal Author(s) Gaver, D. P. and Jacobs, P. A.

ype of Report nnical	13b Time Covered From To	14 Date of Report *(year, month,day)* 1990, August	15 Page Count

pplementary Notation The views expressed in this paper are those of the author and do not reflect the official
/ or position of the Department of Defense or the U.S. Government.

sati Codes		18 Subject Terms *(continue on reverse if necessary and identify by block number)*
Group	Subgroup	Random shapes; Ornstein-Uhlenbeck process; symmetric stable law; Brownian Bridge

ostract *(continue on reverse if necessary and identify by block number*

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istribution/Availability of Abstract unclassified/unlimited ☐ same as report ☐ DTIC users	21 Abstract Security Classification Unclassified
ame of Responsible Individual Gaver	22b Telephone *(Include Area code)* (408) 646-2605 — 22c Office Symbol OR/Gv, OR/Jc

ORM 1473, 84 MAR — 83 APR edition may be used until exhausted — security classification of this page

All other editions are obsolete — Unclassified

MODELING AND SIMULATION OF RANDOM SHAPES BY SCULPTURED WRAPAROUND: PRELIMINARY REPORT

Donald P. Gaver
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1. Introduction

Physical objects with boundaries that are irregular--*random shapes*--occur in profusion in the natural world. Geographical examples range from the continents to small islands, from oceans to lakes to ponds, and from massive ice-covered regions in the Arctic and Antarctic to individual ice floes and icebergs, and their complements, the open spaces or **leads** in ice fields. There are also important random shapes of biological and medical interest and concern. Those who do remote sensing, either of the atmosphere (clouds), the earth or a planet, from satellites will be interested in describing the shapes observed. Those interested in outlining the shapes of mineral or petroleum deposits encounter also the problem of describing the objects of their search. We propose here some possible alternatives.

It appears that there are relatively few mathematical models for flexibly describing such random shapes. The literature is scattered, and no attempt will be made here to cover it. The writer has examined books by Solomon (1978) and Ripley (1981), but the present focus is different. The work of Kendall and his co-workers may be relevant, cf. Kendall (1985,1989).

Our primary objective at this time is to suggest and explore relatively simple ways of modeling or simulating random shapes that resemble those that occur in nature. Another objective is to describe such shapes as they may result from an image processing or filtering algorithm's treatment of the basic shape plus a characterization of a noise. Very few detailed analytical results are provided here. Rather, there will be proposals for future work to expand understanding of the basic approaches to be described.

2. Random Shapes by Perturbation of Regular Shapes

Many of the two-dimensional irregularly shaped bodies encountered in nature are rather regular or even symmetrical in general outline but possessed of irregular boundaries. We proceed to simulate such figures by starting with a circular disk-like central figure and adding to its perimeter a random process. As will be clear from Figures 1-2, the results are of an interesting qualitative character; that character can be easily changed by manipulating a few parameters. Furthermore, analytical facts can, in various cases, be deduced about features of the shapes so generated. There remains much more to be done in that direction, however.

Example 1: Shapes with Circular Centrality and Ornstein-Uhlenbeck Boundaries

An especially simple version of our general approach is to represent the boundary of a two-dimensional random shape as follows in polar coordinates:

$$\begin{aligned} \theta(t) &= t, \qquad 0 \le t \le 2\pi \\ R(t) &= r + f(X(t)) \end{aligned} \tag{1}$$

where X(t) is an Ornstein-Uhlenbeck (diffusion) process

$$dX(t) = -\alpha X(t) + \sigma dW(t), \quad |\alpha| < 1, \tag{2}$$

with {dW(t)} the increment to a standard Wiener process, and $f(x) \geq 0$ for any real x. Useful versions of f are $f(x) = e^{kx}$ and $f(x) = x^2$. These guarantee that the

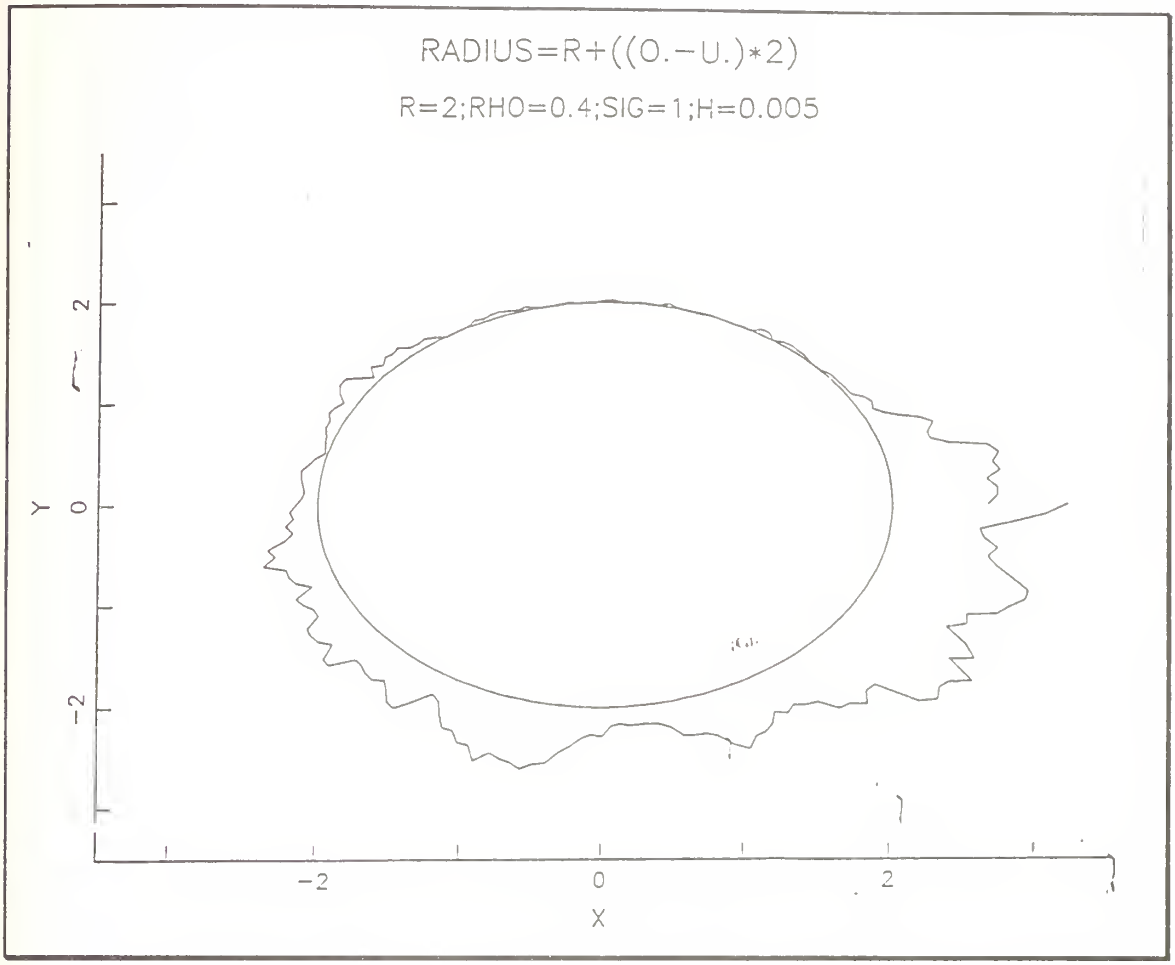

Figure 1. $r = 2$, $f(x) = x^2$; the parameters of the O.U. process are $\sigma = 1.0$, $\alpha = 0.4$

perimeter of the random shape is always outside that of the regular shape underlying the process, namely that of a circular disk of radius r. See Figures 1 and 2 for various realizations of the above; they are of course generated by discretizing time (= angle θ) and replacing X(t) by an equivalent AR–1 process; the process then starts at $t \equiv \theta = 0$ with a random draw from a distribution resembling the O.–U. stationary distribution and is plotted going counterclockwise around to $t \equiv \theta = 2\pi$. The ellipse in each figure is the graph of a circle of radius 2 for the scale of the figure. Note that there is an anticipated (small) jump at the seam at $t = 0 = 2\pi$ which can be mended in various plausible ways. One such is to force the random boundary function $X(t)$ to be a *bridge*, so $X(0) = X(2\pi)$. This is aesthetically pleasing, but probably unnecessary from a practical viewpoint. It seems natural to call the construction suggested a wraparound process.

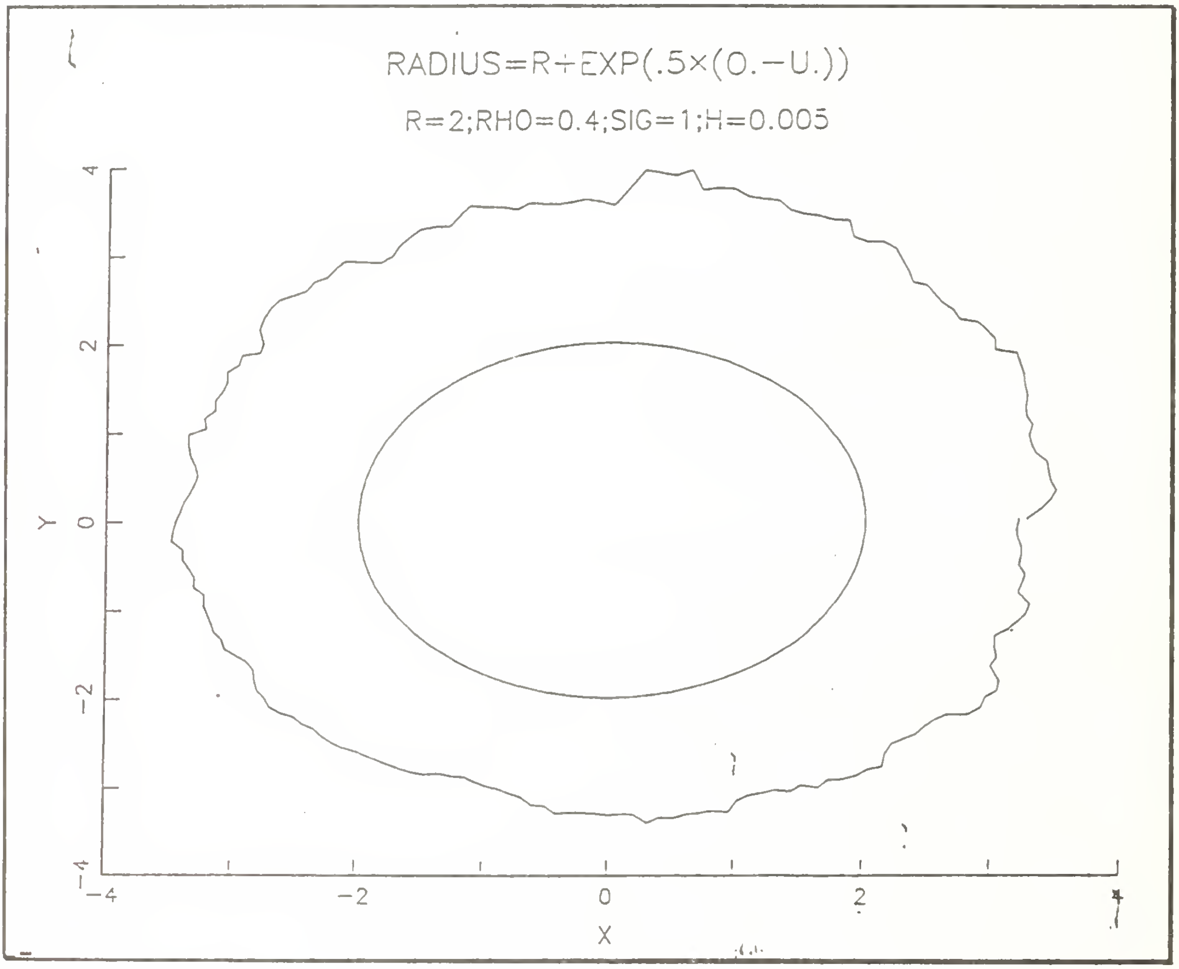

Figure 2. $r = 2$, $f(x) = \exp\left(\frac{1}{2}x\right)$; the parameters of the O.U. process are $\sigma = 1.0$, $\alpha = 0.4$

Certain analytical results are available directly; others will require more careful attention. For example, let A be the area of the random shape generated as described. Then, by polar coordinate rules

$$A = \int_0^{2\pi} \left(r + f(X(t))\right)^2 \frac{1}{2} dt$$

$$= \pi r^2 + r \int_0^{2\pi} f(X(t)) dt + \frac{1}{2} \int_0^{2\pi} f(X(t))^2 dt. \qquad (3)$$

Now take expectations to find

$$E[A] = \pi r^2 + r \int_0^{2\pi} E[f(X(t))] dt + \frac{1}{2} \int_0^{2\pi} E\left[f(X(t))^2\right] dt. \qquad (4)$$

Suppose f(x) = e^{kx}; then if X(t) is O.U. as in (2) we know that

$$X(t) \sim \aleph\left(X(0)e^{-\alpha t}, \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right) \qquad (5)$$

so

$$E\left[e^{+kX(t)}\right] = \exp\left[kX(0)e^{-\alpha t} + \frac{k^2}{2} \cdot \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right].$$

Suppose $X(0) \sim \aleph\left(0, \frac{\sigma^2}{2\alpha}\right)$, the stationary distribution. Then, if so, the unconditional distribution of $X(t) \sim \aleph\left(0, \frac{\sigma^2}{2\alpha}\right)$ independent of t for all t, and we can easily evaluate the mean of the area, A, for any f:

$$E[A] = \pi r^2 + 2\pi r \left\{E[f(X)] + \frac{1}{2} E\left[f^2(X)\right]\right\}. \qquad (6)$$

For f(x) = e^{kx} we get

$$E[A] = \pi r^2 + 2\pi r\left\{e^{k^2\sigma^2/4\alpha} + \frac{1}{2}e^{k^2\sigma^2/\alpha}\right\} \tag{7}$$

which shows that as the correlation coefficient α approaches zero, and the boundaries become exceedingly jagged, the area tends to increase at an exponential rate. Likewise, if C is a random chord that crosses through the center of the figure then

$$C = 2r + f(X(t)) + f(X(t+\pi)) \tag{8}$$

and if X(t) is given the stationary distribution then

$$E[C] = 2r + 2E[f(X)], \tag{9}$$

which in the case $f(x) = e^{kx}$ yields

$$E[C] = 2r + 2e^{k^2\sigma^2/4\alpha} \tag{10}$$

which again suggests the increase in dimension of the shape as α tends to zero.

Covariances of radii can be obtained analytically, and from these the variance of an area. It is also of interest to ask for the expected length of the maximum central chord.

Example 2. The previous process, summed.

There is nothing to prevent one from summing or convex-combining several contributions to the boundary:

$$R(t) = r + \Sigma f_i(X_i(t)) \tag{11}$$

where, for example, $X_i(t) \sim$ O.U. (α_i, σ_i) means that the component processes are Ornstein-Uhlenbeck with their individual parameters. Likewise, the correlation and innovation variance parameters can themselves be, minimally, time-dependent, and possibly even governed by driving stochastic processes, optionally taken to be of the forms

$$\alpha_i(t) = e^{Y_i(t)} / 1 + e^{Y_i(t)} \qquad \text{(Logistic)} \qquad (12)$$

and

$$\sigma_i(t) = e^{Z_i(t)} \qquad \text{(Log - Linear)} \qquad (13)$$

with $Y_i(t)$ and $Z_i(t)$ themselves taken to be appropriate stochastic processes, not necessarily independent and, most conveniently but not necessarily Gaussian. There are many options here, including that of allowing the radius of the basic circular disk to be itself a random process that, perhaps, tends to grow or shrink in time. A candidate for r(t) is the radial Brownian motion or Bessel process; see Feller (1966) Chap. X, Section 6, although there are a great many other options. The O.-U. is illustrative and convenient but certainly not intended to be exclusively, if ever, entirely appropriate.

Example 3: Perturbing Natural Shapes

It may be of interest to start with a natural shape, e.g., an ice floe or atmospheric cloud image, and perturb it by a known process, such as the log-O.-U. This could well represent the appearance of the image of the same entity at a different time point, e.g., at a time when a satellite next passes; the difference in apparent shape could be the result of natural processes plus noise.

Suppose a group of such entities exists, each one of which is imaged at time t, again at $t+\tau$ at which time shape has been randomly modified as described above, ... and so on with another image take at $t+k\tau$ with further random perturbation, at each time. Then an important problem is to identify and track each member of the group over time, under the handicap that shape change can hamper identifiability across different satellite passes. It is proposed to study the above problem, and to investigate the manner in which item identifiability degenerates as the variation in the perturbing boundary process increases.

Note that it is possible to *estimate* the parameters of the perturbing boundary process f(X(t)) by a) first smoothing the figure's perimeter, perhaps by the techniques of Banfield and Raftery (1989), and b) modeling the residual process: the difference of the real/true perimeter and the smoothed perimeter; that difference should be taken orthogonally to the smooth.

Comment: Relation to "Sculpturing"

It was Tukey who originally suggested simple ways of lengthening (or shortening) the tails of distributions such as the Gaussian: if Z is $\aleph(0,1)$ then

$$Y = Ze^{hZ}, \quad h > 0 \tag{14}$$

has zero mean but larger variance owing to the extended tail imposed by e^{hz}. If $h < 0$ then the tail of the distribution of Y is shortened. Such an idea could be applied point-wise to X(t), e.g., O.-U. from (2) to generate a controllable-shaped boundary: replace $X(\theta)$ by $Y(\theta)\equiv Y(t) = X(t)\exp[h\cdot X(t)]$. One could even allow the sculpturing parameter h to be angle-dependent, e.g., let $h(\theta) = -0.2$ for $0\leq\theta<\pi$, $h(\theta) = +0.2$ for $\pi\leq\theta\leq 2\pi$ should result in a figure or shape that is flatter on one side (between 0 and π) and more jagged on the other (between π and 2π). There is no trouble at all to allow the sculpturing parameter h to be itself a random process i.e., replace constant h by $H(\theta)\equiv H(t)$. Furthermore, summary statistics are relatively easy to compute if X(0) is sampled from the stationary distribution of $\{X(t)\}$, presuming the latter to exist. Note that the sculptured figures are exceedingly easy to simulate on a computer: simply generate X(t), and then multiply by exp[hX(t)] to obtain the boundary value at $t = \theta$. [The result is a **sculptured wraparound process** that may be induced to take on natural-appearing shapes by changing a few parameters.]

For an account of general random variable sculpturing see Gaver, et al., (1979) and Gaver (1983). Here we are actually sculpturing a figure or shape, and by a stochastic process realization.

Application

A report by Banfield and Raftery (1989) suggests application of a non-parametric smoothing technique to better delineate, and identify ice floes. Different satellite images of the same ice floe may differ somewhat, and the B-R procedure should be of help in identifying floes (that have not broken up or experienced edge-alteration) as the same when depicted by different images. Their examples suggest that the procedure works well.

It may be of help to check the effectiveness of B-R smoothing when perturbation or sculpturing of a known principal curve (e.g., circle; or one actually derived from real data) with a boundary or perimeter of known stochastic structure is in place. One simple mechanism for generating such a boundary is to utilize our procedure described above. In order to determine the nature of the parameter for the perturbing process f(X(t)) an examination of residuals (Tukey's "rough") obtained from the estimated principal curve (Tukey's "smooth") should be of use. It is our proposal to attempt such an investigation. It may also be found that procedures less computer-intensive than those of B-R can be nearly as effective.

A general question of applied interest might be the use of remote sensing (by satellite) of ice floes to infer forces causing their (relative) motion, that is, the magnitude of winds or currents between successive satellite passes could account for the changes of relative ice floe positions. Consequently, study of position changes of ice floes might provide a remote inference concerning near-surfaces forces. For this to work, identifiability of individual ice floes seems of considerable importance. Such an approach, to infer surface wind speed from satellite sensed white-cap cover, has been taken by Monahan and O'Muircheartaigh (1980).

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APPENDIX A. GENERALIZING THE ORNSTEIN-UHLENBECK PROCESS

The purpose of this note is to examine a model analogous to the Ornstein-Uhlenbeck (O.-U.) process, in which innovations are allowed to have *symmetric stable laws*, cf. Feller (1968), rather than be simply Gaussianly distributed bits of standard Wiener processes. In particular, such processes can have Cauchy innovations. The purpose of this is to introduce processes that give rise to long tails that can provide more adjustable behavior for image boundaries.

Thus for arbitrarily small h > 0 we write

$$X(t+h) = X(t) - \rho h X(t) + \tau h \varepsilon_t \qquad (A-1)$$

where ε_t has, concretely, the Cauchy density

$$\varepsilon_t \sim \frac{1}{\pi}\left(\frac{1}{1+x^2}\right). \qquad (A-2)$$

For this distribution the characteristic function (ch. fcn.) is

$$E\left[e^{i\theta\varepsilon_t}\right] = \bar{e}^{|\theta|}. \qquad (A-3)$$

In the most general case, that of a symmetric stable law,

$$E\left[e^{i\theta\varepsilon_t}\right] = \bar{e}^{|\theta|^{\alpha}}, 1 \le \alpha \le 2; \qquad (A-4)$$

if $\alpha=2$ it is recognized that innovations are Gaussian.

Put

$$\varphi(\theta,t) = E\left[e^{i\theta X(t)}\right] \qquad (A-5)$$

the ch. fcn. of X(t); we keep implicit the initial condition X(0). In general θ must be a real number in order that the rhs of (A-4) be finite. Derive an equation for φ as follows. Write

$$E\left[e^{i\theta X(t+h)}\right] = E\left[e^{i\theta\{X(t)(1-\rho h)+\tau h\varepsilon_t\}}\right]$$
$$= E\left[e^{i\theta\{X(t)(1-\rho h)\}}\right]E\left[e^{i\theta\tau h\varepsilon_t}\right] = \varphi(\theta(1-\rho h),t)e^{-h\xi|\theta|^\alpha} \qquad (A-6)$$

by the assumed independence of successive innovations; note that it is necessary to scale innovations so that $\tau^\alpha h^\alpha = \xi h$ for fixed ξ, or $\tau = \xi^{1/\alpha}h^{(1-\alpha)/\alpha}$ for fixed ξ. If h becomes small,

$$E\left[e^{i\theta X(t+h)}\right] \equiv \varphi(\theta,t+h) \approx \left[\varphi(\theta,t) + \left(\frac{\partial}{\partial\theta}\varphi(\theta,t)\right)(-\rho\theta)h + O\left(h^2\right)\right]\left[1 - h\xi|\theta|^\alpha + O\left(h^2\right)\right] \quad (A-7)$$

This done, then

$$\frac{\varphi(\theta,t+h)-\varphi(\theta,t)}{h} = -\rho\theta\frac{\partial\varphi}{\partial\theta} - \xi|\theta|^\alpha\varphi + O(h) \qquad (A-8)$$

and if h→0 the following quasi-linear first-order partial differential equation results:

$$\frac{\partial\varphi}{\partial t} + \rho\theta\frac{\partial\varphi}{\partial\theta} = -\xi|\theta|^\alpha\varphi. \qquad (A-9)$$

Steady-State or Long-Run Solution

Suppose

$$\lim_{t\to\infty}\varphi(\theta,t) = \varphi^{\#}(\theta) \qquad (A-10)$$

is a bona-fide ch. fcn. of an honest distribution. This is at least plausible since ρ>0 and there is a restoring force tendency in the dynamics of (A-1). We then have the ordinary first-order differential equation

$$\rho\theta\frac{d\varphi^{\#}}{d\theta} = -\xi|\theta|^{\alpha}\varphi^{\#} \qquad (A-11)$$

or

$$\frac{d\varphi^{\#}}{\varphi^{\#}} = -\frac{\xi}{\rho}\frac{|\theta|}{\theta}\cdot|\theta|^{\alpha-1}d\theta = -\frac{\xi}{\rho}s(\theta)|\theta|^{\alpha-1}d\theta \qquad (A-12)$$

where

$$s(\theta) = \begin{cases} +1 & \text{if } \theta \geq 0 \\ -1 & \text{if } \theta < 0 \end{cases}. \qquad (A-13)$$

Write (A-12) as

$$\frac{d\varphi^{\#}}{\varphi^{\#}} = -\frac{\xi}{\rho}|\theta|^{\alpha-1}\, d|\theta|$$

and integrate to obtain

$$\varphi^{\#}(\theta) = \exp\left[-\frac{\xi}{\rho\alpha}|\theta|^{\alpha}\right]. \qquad (A-14)$$

This is the ch.fcn. of a symmetric stable law, as before. If $\alpha=2$, $\varphi^{\#}$ is the ch. fcn. of a Gaussian/Normal with variance ξ/ρ, as is used earlier, (2). If $\alpha=1$, $\varphi^{\#}$ represents a Cauchy distribution with scale parameter ξ/ρ. Other special cases may exist, but are far less familiar and convenient.

Transient Solution

The equation (A-9) can be solved by the standard method of characteristics; see Street (1973). Another approach that reveals the underlying structure of the process is this: express (A-1) as

$$dX(t) = -\rho X(t)dt + \tau\varepsilon_t dt \qquad (A-15)$$

or formally as

$$\frac{dX}{dt} = -\rho X(t) + \tau\varepsilon_t. \qquad (A-16)$$

Note that we can allow ρ to depend on t at no cost:

$$\frac{dX}{dt} = -\rho(t)X(t) + \tau\varepsilon_t. \qquad (A-17)$$

Use the standard integrating factor:

$$\frac{d}{dt}\left[X(t)e^{\int_0^t \rho(x)dx}\right] = \tau\varepsilon_t e^{\int_0^t \rho(x)dx}; \qquad (A-18)$$

so if $X(0) = X_0$,

$$X(t) = X_0 e^{\int_0^t \rho(x)dx} + \int_0^t e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy. \qquad (A-19)$$

Clearly X(t) has the structure of a weighted sum of components; a typical member of the integrand being

$$e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy. \qquad (A-20)$$

Therefore

$$E\left[e^{i\theta X(t)}\right] = E\left[\exp\left\{i\theta\left[X_0 e^{\int_0^t \rho(x)dx} + \int_0^t e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy\right]\right\}\right]$$

$$= E\left[\exp\left\{i\theta X_0 e^{\int_0^t \rho(x)dx}\right\}\right] \cdot E\left[\exp\left\{i\theta \int_0^t e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy\right\}\right] \qquad (A-21)$$

Now think of the integral $\int_0^t e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy$ as approximated by a sum of elements, a typical value being (A-20). Use the independence of successive ε_y contributions to obtain

$$E\left[\exp\left\{i\theta\int_0^t e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy\right\}\right] = \exp\left\{-|\theta|^\alpha \int_0^t e^{-\alpha\int_y^t \rho(x)dx} \xi dy\right\} \qquad (A-22)$$

for, by the fact that ε_y obeys a stable law

$$E\left[\exp\left\{i\theta e^{-\int_y^t \rho(x)dx} \tau\varepsilon_y dy\right\}\right] = \exp\left\{-|\theta|^\alpha e^{-\alpha\int_y^t \rho(x)dx} \tau^\alpha (dy)^\alpha\right\} \qquad (A-23)$$

again we have put $\xi dy = \tau^\alpha (dy)^\alpha$ just as was done before. It follows that the solution to (A-9) must be

$$\varphi(\theta,t) = \exp\left\{i\theta X_0 e^{-\rho t} - |\theta|^\alpha \frac{\xi}{\alpha\rho}\left(1-e^{-\alpha\rho t}\right)\right\} \qquad (A-24)$$

Thus for fixed t,

$$X(t) \sim \text{Stable}\left(\text{center} = X_0 e^{-\rho t}, \text{scale} = \left(\frac{\xi}{\alpha\rho}\left(1-e^{-\alpha\rho t}\right)\right)^{\frac{1}{\alpha}}; \text{index} = \alpha\right); \quad (A-25)$$

the center is the mean and the scale is the standard deviation in the Gauss/Normal case $\alpha = 2$. Of course (A-25) agrees with (A-14) as $t \to \infty$, and differentiation will show that (A-24) solves (A-9). *Note*: the rate of approach to steady state, $\alpha\rho$, increases with α, so the Gauss/Normal case (classical O.-U.) approaches steady-state more quickly than does the analogous Cauchy-driven O.-U. with same ρ.

Simulation

In order to simulate X(t) it will be necessary to discretize time into, say, h-increments; the result will be a path or trajectory $X_h(t)$. In order to obtain the desired marginal distributions, e.g., (A-25) simply use (A-1), i.e.

$$X(t+h) = X(t) - \rho h X(t) + \tau h \varepsilon_t, \qquad t = 0, h, 2h, \ldots$$

where $\tau = \xi^{1/\alpha} h^{(1-\alpha)/\alpha}$ and ε_t is an α-stable random variable with unit scale. For example(s):

$\alpha = 2$; ε_t is a draw from $e^{0.5x^2} / \sqrt{2\pi}$ (Normal density)

$\alpha = 1$; ε_t is a drawn from $\left[\pi\left(1+x^2\right)\right]^{-1}$ (Cauchy density).

For fixed ξ and ρ the paths of the Cauchy O.-U. should tend to exhibit more long-range order than those of the Gauss O.-U. The option of choosing ρ=ρ(t) allows further freedom in adjusting path behavior.

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APPENDIX B. BRIDGE PROCESSES

One feature common to Figures 1 and 2 is the jump that appears at the seam of $t=0=2\pi$. The purpose of this Appendix is to describe one procedure to obtain shapes that are "seamless." To this end processes for the Ornstein-Uhlenbeck (O.-U.) and Cauchy autoregressive processes are introduced that correspond to the Brownian Bridge.

Let $\{W(t);\ 0 \le t \le 1\}$ be a standard Brownian motion process with $W(0)=0$. Recall that a Brownian Bridge can be obtained by the transformation

$$B(t) = W(t) - tW(1), \quad 0 \le t \le 1;$$

note that $B(1) = B(0) = 0$; cf. Karlin and Taylor (1980).

In a similar spirit let $\{X(t),\ t \ge 0\}$ be an O.-U. process or a generalization of it as described in Appendix A with $X(0) = 0$. We will define a Bridge process as

$$U(t) = X(t) - tX(1), \text{ for } 0 \le t \le 1.$$

Note that $X(0) = X(1) = 0$. One way not to have $X(0) = X(1) = 0$ is to add a random variable U° to $U(t)$ having the stationary distribution of $\{X(t);\ t \ge 0\}$.

Let $\{X(t);\ t \ge 0\}$ be an O.-U. process with drift coefficient $-\rho$, infinitesimal variance σ^2, and $X(0) = 0$. Set

$$Y(t) = X(t) - tX(1), \quad 0 \le t \le 1.$$

$\{Y(t);\ t \ge 0\}$ is a Gaussian process with mean 0. Further, for $s<t$

$$\begin{aligned} E[Y(t)Y(s)] &= E[(X(t) - tX(1))(X(s) - sX(1))] \\ &= E[X(t)X(s)] - tE[X(1)X(s)] - sE[X(t)X(1)] + tsE\left[X(1)^2\right]. \\ &= \frac{\sigma^2}{2\rho}\left\{e^{-\rho(t+s)}\left[e^{2\rho s} - 1\right] - te^{-\rho(1+s)}\left[e^{2\rho s} - 1\right] - se^{-\rho(1+t)}\left[e^{2\rho t} - 1\right] + tse^{-2\rho}\left[e^{2\rho} - 1\right]\right\}. \end{aligned}$$

Thus,

$$E\left[Y(t)^2\right] = \text{Var}\left[Y(t)\right]$$
$$= \frac{\sigma^2}{2\rho}\left[\left(1-e^{-2\rho t}\right) - 2te^{-\rho(1+t)}\left(e^{2\rho t}-1\right) + t^2\left(1-e^{-2\rho}\right)\right].$$

Simulation.

The O.-U. Bridge can be simulated as follows. X° is generated from a normal distribution having mean 0 and variance $\sigma^2/2\rho$. The Ornstein-Uhlenbeck process is simulated as X(0)=0

$$X(t+h) = X(t) - \rho h X(t) + \sqrt{h}\,\varepsilon_t$$

where $\{\varepsilon_t\}$ is a sequence of independent normal random variables with mean 0 and variance σ^2. A bridge is created by calculating

$$\tilde{U}(t) = X(t) - tX(1) \quad 0 \le t \le 1.$$

The random variable X° is added to $\tilde{U}(t)$ to obtain the process

$$U(t) = \tilde{U}(t) + X°.$$

In the figure 1B, the radius of the shape at time $2\pi t$ is taken to be

$$R + |U(t)|.$$

The parameters of the simulation are R = 2, ρ = 0.4, σ = 1 and h = .005. The ellipse in the center of the figure is the circle of radius 2 for the scale of the picture.

The Cauchy Bridge is simulated as follows. X° is generated from a Cauchy distribution having density function

$$f^{\circ}(y) = (\xi / \rho)^{-1} \frac{1}{\pi}\left[1 + (y\rho / \xi)^2\right]^{-1}.$$

In the simulation,

$$X(0) = 0$$
$$X(t+h) = X(t) - \rho h X(t) + \xi \varepsilon(t) h \quad 0 \le t \le 1$$

where $\{\varepsilon(t)\}$ is a sequence of independent standard Cauchy random variables. A Bridge is created by calculating

$$\tilde{U}(t) = X(t) - tX(1). \quad 0 \le t \le 1.$$

Finally, a Bridge not tied down at 0 is obtained by setting

$$U(t) = \tilde{U}(t) + X^{\circ}.$$

In Figure 2B the radius of the shape at time $2\pi t$ is taken to be

$$R + |U(t)|.$$

The parameters of the simulation are R = 2, ρ = 0.4, h = 0.005, and ξ = 0.8. Once again the ellipse in the center is the circle of radius 2 for the scale of the figure.

Reference

S. Karlin and H. M. Taylor (1981) *A Second Course in Stochastic Processes*. Academic Press Inc., New York.

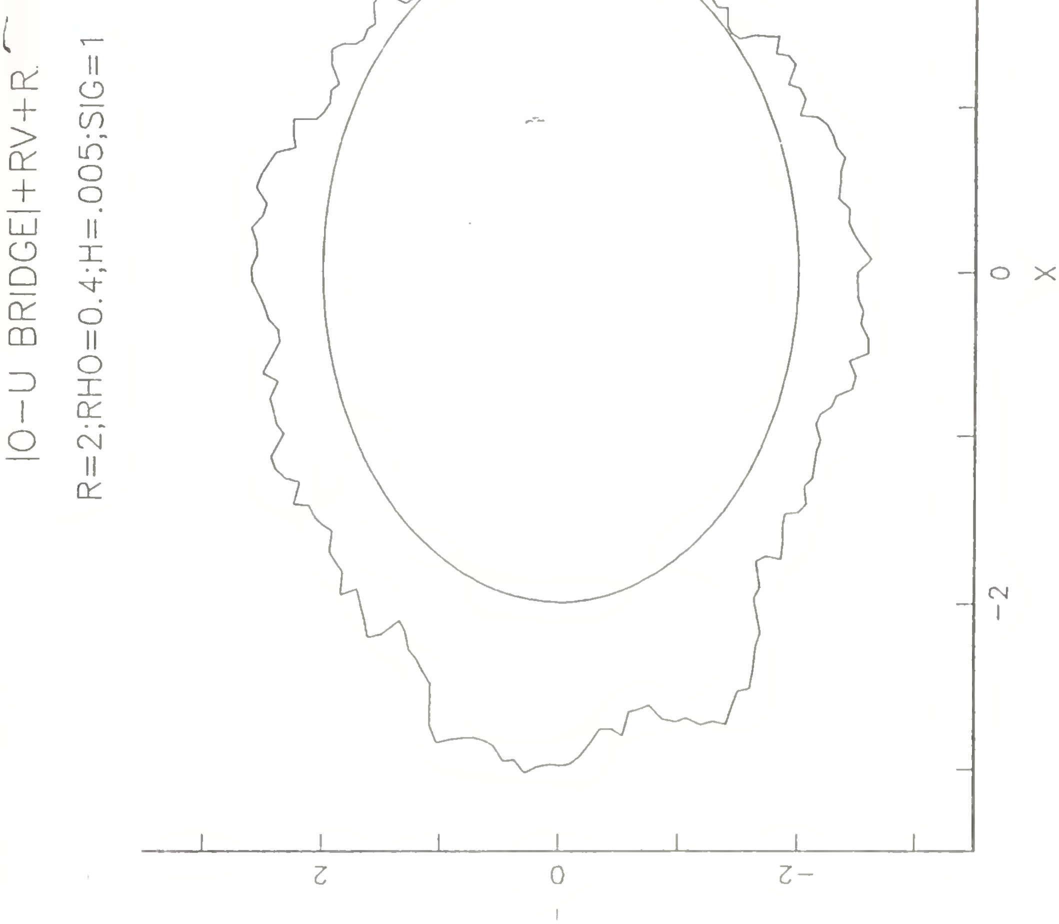

Figure 1B

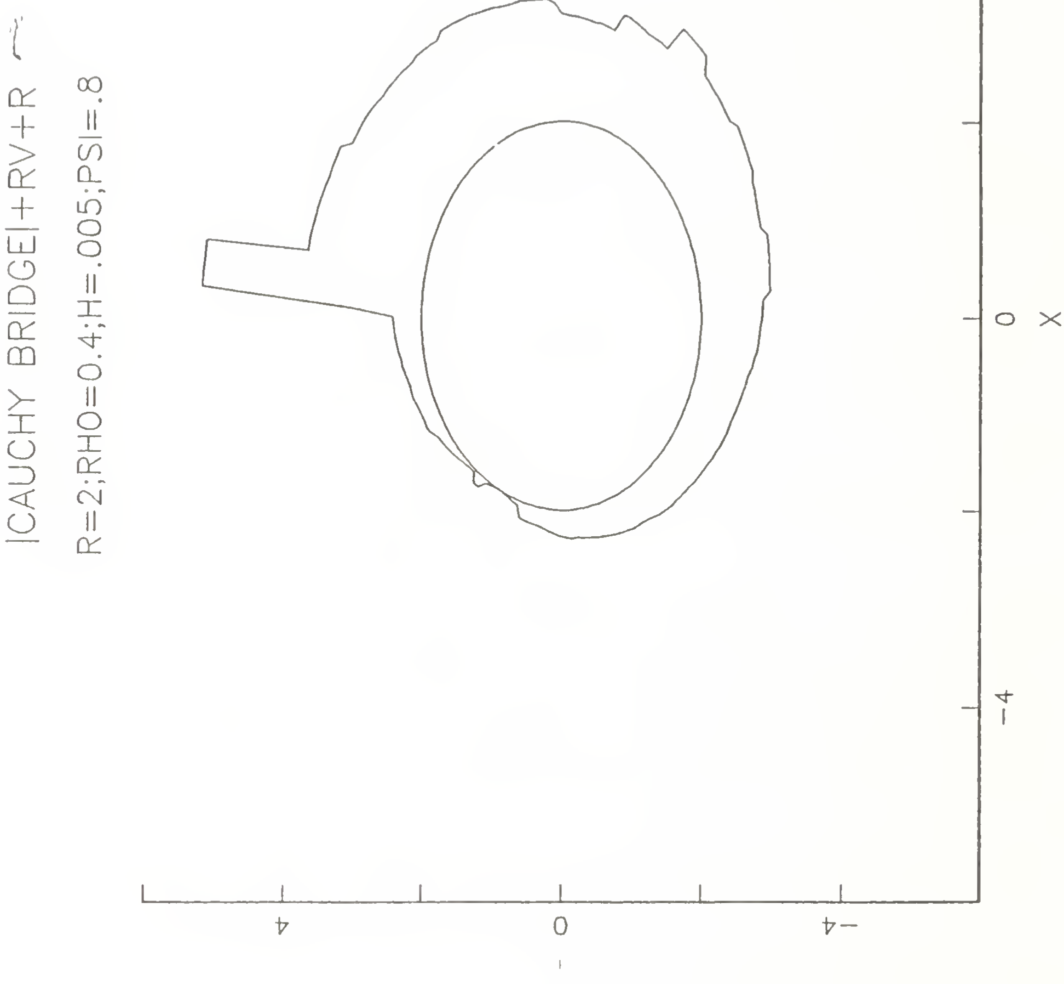

Figure 2B

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